A SIMPLE COUNTEREXAMPLE TO THE JORDAN–HÖLDER PROPERTY FOR DERIVED CATEGORIES

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Abstract. A counterexample to the Jordan–Hölder property for semiorthogonal decompositions of derived categories of smooth projective varieties was constructed by Böhning, Graf von Bothmer and Sosna. In this short note we present a simpler example by realizing Bondal's quiver in the derived category of a blowup of $\mathbb{P}^3$.

1. Introduction

Given a triangulated category $\mathcal{T}$, a semiorthogonal decomposition for $\mathcal{T}$ is a chain $\mathcal{A}_1, \dots, \mathcal{A}_m \subset \mathcal{T}$ of full triangulated subcategories such that

- for any $j > i$ one has $\mathsf{Hom}(\mathcal{A}_j, \mathcal{A}_i) = 0$, and
- for any object $T \in \mathcal{T}$ there is a chain of morphisms

$$0 = T_m \to T_{m-1} \to \dots \to T_1 \to T_0 = T$$

such that $\mathsf{Cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$.

We write $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ to denote a semiorthogonal decomposition.

It is well known [BK] that the braid group acts on the set of all semiorthogonal decompositions of a given category — the i-th generator of the braid group acts as

$$\langle \mathcal{A}_1, \dots, \mathcal{A}_r \rangle \mapsto \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, {}^{\perp}\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1} \rangle \cap \langle \mathcal{A}_{i+2}, \dots, \mathcal{A}_m \rangle^{\perp}, \mathcal{A}_{i+2}, \dots, \mathcal{A}_m \rangle.$$

So if a category $\mathcal{T}$ has a semiorthogonal decomposition, it has many of them. However, it is also well known that the equivalence classes of the components do not change under this action — there is an equivalence of categories

$${}^{\perp}\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1} \rangle \cap \langle \mathcal{A}_{i+2}, \dots, \mathcal{A}_r \rangle^{\perp} \cong \mathcal{A}_i$$

This motivates the following definition.

Definition 1.1. A triangulated category $\mathcal{T}$ has the Jordan–Hölder property if for any pair

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle, \qquad \mathcal{T} = \langle \mathcal{B}_1, \dots, \mathcal{B}_n \rangle$$

of semiorthogonal decompositions with indecomposable components one has $m = n$ and there is a permutation $\sigma \in S_m$ such that $\mathcal{B}_i \cong \mathcal{A}_{\sigma(i)}$ for each $1 \le i \le m$.

Among triangulated categories of geometrical nature there are very few for which the Jordan-Hölder property has been proved. Basically these consist of those for which all semiorthogonal decompositions can be classified (basically these are $\mathbf{D}(\mathbb{P}^1)$ and its quotient stacks $\mathbf{D}(\mathbb{P}^1/\Gamma)$, see [Kir] for the detailed investigation of the latter), or those which are themselves indecomposable (connected Calabi–Yau categories [Bri], derived categories of curves of positive genus [Oka]). Even for $\mathbb{P}^2$ the property is questionable.

On one hand, if the Jordan–Hölder property could be justified for derived categories of smooth projective varieties, this would allow to define nice birational invariants, see [Kuz] and [BBS]. On the other

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hand, it was known for a long time that the property is not satisfied for arbitrary triangulated categories. A very simple counterexample was constructed by Alexei Bondal long ago. Namely, Bondal considered a quiver with relations

$$Q = \left(\bullet \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \bullet \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \bullet \;\middle|\; \beta_1\alpha_2 = \beta_2\alpha_1 = 0 \right) \tag{1}$$

and noted that on one hand as any oriented quiver it has a full exceptional collection

$$\mathbf{D}(Q) = \langle P_1, P_2, P_3 \rangle$$

with P_i being the projective module of the i-th vertex, and on the other hand, it has an exceptional object

$$P = \left(\mathsf{k} \underset{0}{\overset{1}{\rightrightarrows}} \mathsf{k} \underset{0}{\overset{1}{\rightrightarrows}} \mathsf{k} \right) \tag{2}$$

which is nonextendable, i.e. does not extend to a longer exceptional collection (even numerically).

Since the category $\mathbf{D}(Q)$ itself is not equivalent to the derived category of a scheme, Bondal's counterexample does not answer the question whether the Jordan–Hölder property is true for derived categories of schemes, so for some time there was a little hope that by some miracle it might be true. A recent paper of Böhning, Graf von Bothmer and Sosna [BBS] gave finally a negative answer to this question. To be more precise, investigating the derived category $\mathbf{D}(X)$ of the classical Godeaux surface

$$X = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\}/\mathbb{Z}_5 \subset \mathbb{P}^3/\mathbb{Z}_5,$$

where $\mathbb{Z}_5$ acts with weight i on x_i, the authors constructed two nonextendable exceptional collections in $\mathbf{D}(X)$, one of length 11 (the maximal possible), and the other of length 9. The nonextendability is also checked on the numerical level. The construction of this counterexample and the proofs are rather complicated up to such extent that at some moment a computer computation (using Macaulay2) is used.

The goal of this note is to give an elementary example. We note that although $\mathbf{D}(Q)$ itself is not equivalent to the derived category of an algebraic variety, it can be realized as a semiorthogonal component of such. And this is enough to get a counterexample. The variety we consider is a two-step blowup of $\mathbb{P}^3$ in two smooth rational curves. Its derived category has a full exceptional collection and we observe that it contains Bondal's quiver as a subcollection.

2. Example

Let $A \subset V$ be a pair of vector spaces of dimensions 2 and 4 respectively, so that $\mathbb{P}(A) \subset \mathbb{P}(V)$ is a line $\mathbb{P}^1$ in a $\mathbb{P}^3$. Let X be the blowup of $\mathbb{P}(V)$ along $\mathbb{P}(A)$ with E being the exceptional divisor. Then

$$E \cong \mathbb{P}(A) \times \mathbb{P}(V/A) = \mathbb{P}^1 \times \mathbb{P}^1.$$

We denote by i the embedding of E into X and by H the pullback to X of a hyperplane on $\mathbb{P}(V)$. The Picard group of X is generated by H and E and we have

$$\mathcal{O}_X(H)_E \cong \mathcal{O}_E(1,0). \tag{3}$$

Let C be a smooth rational curve on X which intersects E transversally in two points

$$P_1 = (a_1, b_1), \qquad P_2 = (a_2, b_2),$$

where $a_i \in \mathbb{P}(A)$, $b_i \in \mathbb{P}(V/A)$ and with

$$a_1 \neq a_2.$$

For example, one can take C to be the proper preimage of a conic in $\mathbb{P}(V)$ intersecting the line $\mathbb{P}(A)$ in two distinct points (in this case the points b_1 and b_2 will coincide, but this does not matter).

Let $\pi : Y \to X$ be the blowup of X in C. Let E' be the exceptional divisor of this blowup, $i' : E' \to Y$ be its embedding into Y, $p : E' \to C$ the projection, and $j : C \to X$ the embedding of the curve. This can be summarized in a diagram

$$\begin{tikzcd} E' \arrow[r, "i'"] \arrow[d, "p"'] & Y \arrow[d, "\pi"] \\ C \arrow[r, "j"] & X \end{tikzcd}$$

Recall that by Orlov's blowup formula [Or] we have a semiorthogonal decomposition

$$\mathbf{D}(Y) = \langle \pi^*(\mathbf{D}(X)), i'_* p^*(\mathbf{D}(C)) \rangle.$$

We take the following triple of sheaves on Y:

$$\mathcal{E}_1 = \mathcal{O}_Y = \pi^*\mathcal{O}_X, \qquad \mathcal{E}_2 = \pi^* i_* \mathcal{O}_E(1,0), \qquad \mathcal{E}_3 = i'_* p^* \mathcal{O}_C(-3).$$

Lemma 2.1. *The triple $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ is exceptional and extends to a full exceptional collection in $\mathbf{D}(Y)$.*

Proof. First, we extend the pair $(\mathcal{O}_X, i_*\mathcal{O}_E(1,0))$ to a full exceptional collection in $\mathbf{D}(X)$:

$$\mathbf{D}(X) = \langle \mathcal{O}_X(-3H), \mathcal{O}_X(-2H), \mathcal{O}_X(-H), \mathcal{O}_X, i_*\mathcal{O}_E, i_*\mathcal{O}_E(1,0) \rangle.$$

This is just the full exceptional collection obtained by combining the standard exceptional collection $(\mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O})$ on $\mathbb{P}(V)$ with the standard collection $(\mathcal{O}, \mathcal{O}(1))$ on $\mathbb{P}(A)$ if we consider X as the blowup of $\mathbb{P}(V)$ in $\mathbb{P}(A)$. Pulling it back to Y and combining with the exceptional collection $\mathcal{O}_C(-3), \mathcal{O}_C(-2)$ on C we obtain a full exceptional collection in $\mathbf{D}(Y)$:

$$\mathbf{D}(Y) = \langle \mathcal{O}_Y(-3H), \mathcal{O}_Y(-2H), \mathcal{O}_Y(-H), \underline{\mathcal{O}_Y}, \pi^* i_*\mathcal{O}_E, \underline{\pi^* i_*\mathcal{O}_E(1,0)}, \underline{i'_* p^*\mathcal{O}_C(-3)}, i'_*\mathcal{O}_C(-2) \rangle$$

(we denote here the pullback of H to Y also by H). The underlined terms of this exceptional collection are the objects $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$. $\square$

Lemma 2.2. *We have*

$$\mathsf{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_2) = A^*, \qquad \mathsf{Ext}^\bullet(\mathcal{E}_2, \mathcal{E}_3) = \mathsf{k}^2[-1], \qquad \mathsf{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_3) = \mathsf{k}^2[-1].$$

Here the brackets stand for the homological shift. In other words, it is claimed that between $\mathcal{E}_1$ and $\mathcal{E}_2$ there is only Hom, while from $\mathcal{E}_1$ and $\mathcal{E}_2$ to $\mathcal{E}_3$ there are only Ext^1.

Proof. Since π^* is fully faithful we have

$$\mathsf{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_2) = \mathsf{Ext}^\bullet(\mathcal{O}_X, i_*\mathcal{O}_E(1,0)) = H^\bullet(E, \mathcal{O}_E(1,0)) = A^*.$$

Furthermore, for any $F \in \mathbf{D}(X)$, $G \in \mathbf{D}(C)$ we have

$$\mathsf{Ext}^\bullet(\pi^*F, i'_*p^*G) \cong \mathsf{Ext}^\bullet(F, \pi_* i'_* p^* G) \cong \mathsf{Ext}^\bullet(F, j_* p_* p^* G) \cong \mathsf{Ext}^\bullet(F, j_* G).$$

It follows that

$$\mathsf{Ext}^\bullet(\mathcal{E}_1, \mathcal{E}_3) = \mathsf{Ext}^\bullet(\mathcal{O}_X, j_*\mathcal{O}_C(-3)) = H^\bullet(C, \mathcal{O}_C(-3)) = \mathsf{k}^2[-1].$$

Finally,

$$\mathsf{Ext}^\bullet(\mathcal{E}_2, \mathcal{E}_3) = \mathsf{Ext}^\bullet(i_*\mathcal{O}_E(1,0), j_*\mathcal{O}_C(-3)).$$

To compute the latter we take the resolution

$$0 \to \mathcal{O}_X(H - E) \xrightarrow{E} \mathcal{O}_X(H) \to i_*\mathcal{O}_E(1,0) \to 0$$

and apply the local $\mathcal{H}om(-, j_*\mathcal{O}_C(-3))$ functor. We deduce that $\mathbf{R}\mathcal{H}om(i_*\mathcal{O}_E(1,0), j_*\mathcal{O}_C(-3))$ is quasi-isomorphic to the complex

$$j_*\mathcal{O}_C(-H - 3p) \xrightarrow{E} j_*\mathcal{O}_C(E - H - 3p),$$

where p stands for the class of a point on C, with terms in grading 0 and 1 respectively. Since C is a smooth curve and E is a section of a line bundle on C vanishing with multiplicity 1 at points P_1 and P_2 only, we see that the above map has trivial kernel and its cokernel is just the sum of $\mathcal{O}_{P_1}$ and $\mathcal{O}_{P_2}$, the structure sheaves of the points. It follows that

$$\mathcal{E}xt^\bullet(i_*\mathcal{O}_E(1,0), j_*\mathcal{O}_C(-3)) \cong \mathcal{O}_{P_1}[-1] \oplus \mathcal{O}_{P_2}[-1].$$

Using the local-to-global spectral sequence we deduce that

$$\mathsf{Ext}^\bullet(i_*\mathcal{O}_E(1,0), j_*\mathcal{O}_C(-3)) = H^\bullet(Y, \mathcal{O}_{P_1})[-1] \oplus H^\bullet(Y, \mathcal{O}_{P_2})[-1]. \tag{4}$$

which gives the last claim of the Lemma. □

It remains to compute the multiplication map. Let α_1, α_2 be the basis of A^* dual to the basis a_1, a_2 of A given by the first coordinates of the points P_1 and P_2. Let β_1, β_2 be the basis of $\mathsf{Ext}^1(\mathcal{E}_2, \mathcal{E}_3)$ given by the spaces $H^0(Y, \mathcal{O}_{P_1})$ and $H^0(Y, \mathcal{O}_{P_2})$ in (4) respectively.

Proposition 2.3. *The multiplication map*

$$m : \mathsf{Hom}(\mathcal{E}_1, \mathcal{E}_2) \otimes \mathsf{Ext}^1(\mathcal{E}_2, \mathcal{E}_3) \to \mathsf{Ext}^1(E_1, E_3)$$

is surjective and its kernel is spanned by $\alpha_1 \otimes \beta_2$ *and* $\alpha_2 \otimes \beta_1$*.*

Before giving a proof let us mention the consequences.

Corollary 2.4. *The algebra of homomorphisms of the exceptional collection* $\mathcal{E}_1, \mathcal{E}_2[1], \mathcal{E}_3[1]$ *is isomorphic to the path algebra of Bondal's quiver* (1)*.*

Corollary 2.5. *The derived category* $\mathbf{D}(Y)$ *of* Y *does not have the Jordan–Hölder property.*

Proof of the Proposition. First, the pullback-pushforward adjunction for the morphism π shows that the map m coincides with the multiplication map

$$\mathsf{Hom}(\mathcal{O}_X, i_*\mathcal{O}_E(1,0)) \otimes \mathsf{Ext}^1(i_*\mathcal{O}_E(1,0), j_*\mathcal{O}_C(-3)) \to \mathsf{Ext}^1(\mathcal{O}_X, j_*\mathcal{O}_C(-3)).$$

Now let us identify the bases in the spaces we are interested in.

The first space $\mathsf{Hom}(\mathcal{O}_X, i_*\mathcal{O}_E(1,0))$ has α_1, α_2 as a base. By (3) we can find sections $\bar{\alpha}_1, \bar{\alpha}_2$ of $\mathcal{O}_X(H)$ which restrict to the sections α_1 and α_2 of $\mathcal{O}_E(1,0)$. These are given just by two planes in $\mathbb{P}^3 = \mathbb{P}(V)$ intersecting the line $\mathbb{P}^1 = \mathbb{P}(A)$ transversally in points a_2 and a_1 respectively.

Further, denote by ρ_i the canonical morphisms

$$\mathcal{O}_E(1,0) \xrightarrow{\rho_i} \mathcal{O}_{P_i}$$

and by η_i the canonical extensions

$$\mathcal{O}_{P_i} \xrightarrow{\eta_i} \mathcal{O}_C(-3)[1].$$

Then

$$\beta_i = j_*(\eta_i) \circ \rho_i.$$

To see this consider (the pushforward to X of) the exact sequence

$$0 \to \mathcal{O}_C(-3) \xrightarrow{P_1+P_2} \mathcal{O}_C(-1) \to \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2} \to 0 \tag{5}$$

corresponding to the sum of extension η_1 and η_2, and apply the functor $\mathsf{Hom}(i_*\mathcal{O}_E(1,0), -)$ to it. We will get an exact sequence

$$\begin{aligned} \dots \to \mathsf{Hom}(i_*\mathcal{O}_E(1,0), j_*\mathcal{O}_C(-1)) \to \\ \to \mathsf{Hom}(i_*\mathcal{O}_E(1,0), \mathcal{O}_{P_1}) \oplus \mathsf{Hom}(i_*\mathcal{O}_E(1,0), \mathcal{O}_{P_2}) \to \\ \to \mathsf{Ext}^1(i_*\mathcal{O}_E(1,0), j_*\mathcal{O}_C(-3)) \to \dots \end{aligned}$$

The first term is zero (this is proved analogously to (4)). Consequently, the second map is an embedding. Clearly, it takes the basis (ρ_1, ρ_2) of the second term to $j_*(\eta_1) \circ \rho_1$ and $j_*(\eta_2) \circ \rho_2$ respectively, which thus span the second space. Clearly, these elements coincide with β_1 and β_2.

Now we can check that the products of $\alpha_1 \otimes \beta_2$ and of $\alpha_2 \otimes \beta_1$ are zero. Indeed,

$$m(\alpha_1 \otimes \beta_2) = \beta_2 \circ \alpha_1 = j_*(\eta_2) \circ \rho_2 \circ \alpha_1$$

and already the composition

$$\rho_2 \circ \alpha_1 : \mathcal{O}_X \xrightarrow{\alpha_1} \mathcal{O}_E(1,0) \xrightarrow{\rho_2} \mathcal{O}_{P_2}$$

is zero since α_1 vanishes at point P_2. The same argument applies to $\alpha_2 \otimes \beta_1$.

So, to finish the proof of the Proposition it remains to check that the products of $\alpha_1 \otimes \beta_1$ and $\alpha_2 \otimes \beta_2$ are linearly independent in the space $\mathsf{Ext}^1(\mathcal{O}_X, j_*\mathcal{O}_C(-3))$. For this we note that

$$m(\alpha_i \otimes \beta_i) = \beta_i \circ \alpha_i = j_*(\eta_i) \circ \rho_i \circ \alpha_i$$

and that the composition

$$\rho_i \circ \alpha_i : \mathcal{O}_X \xrightarrow{\alpha_i} \mathcal{O}_E(1,0) \xrightarrow{\rho_i} \mathcal{O}_{P_i}$$

is equal to the canonical evaluation map $e_i : \mathcal{O}_X \to \mathcal{O}_{P_i}$. Finally, applying the functor $\mathsf{Hom}(\mathcal{O}_X, -)$ to sequence (5), we obtain an exact sequence

$$\dots \to \mathsf{Hom}(\mathcal{O}_X, j_*\mathcal{O}_C(-1)) \to \mathsf{Hom}(\mathcal{O}_X, \mathcal{O}_{P_1}) \oplus \mathsf{Hom}(\mathcal{O}_X, \mathcal{O}_{P_2}) \to \mathsf{Ext}^1(\mathcal{O}_X, j_*\mathcal{O}_C(-3)) \to \dots$$

Again, its first term is zero since the line bundle $\mathcal{O}_C(-1)$ on the rational curve C is acyclic, hence the second map is an embedding. This means that the images $j_*(\eta_i) \circ e_i$ of the canonical evaluation maps e_i are linearly independent in $\mathsf{Ext}^1(\mathcal{O}_X, j_*\mathcal{O}_C(-3))$. This precisely means that $m(\alpha_i \otimes \beta_i)$ are linearly independent. □

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